Differentiability, norms, and p-variation

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The talk will survey some matters I've worked on at different times, including selected material from a forthcoming monograph with R. Norvaiša [1].

One thesis, coming from work on empirical processes, is that there is a great variety of norms available, from which one can choose in proving Fréchet differentiability of some functionals in a way that fits with uniform central limit theorems. On domains with dimension larger than 1, one may use norms based on Vapnik-Červonenkis classes of sets. For one-dimensional (time) domains, but where the range may be infinite-dimensional, the joint work mentioned has especially treated p-variation norms.

In very interesting and innovative work over the past 20 years, e.g. [3], Terry Lyons and co-workers consider p-variation for continuous, nondifferentiable paths such as those of diffusion processes, with special treatment for $p \geq 2$. Here, however, p-variation will be taken in the simpler, direct sense as defined before the Lyons et al. work. Jump discontinuities will be not only allowed but essential.

The classical empirical process $\alpha_n = \sqrt{n}(F_n - F)$ converges in p-variation norm to a Brownian bridge [2] only for $p > 2$. Nevertheless the growth of the p-variation of the empirical process for $p < 2$ is at a known rate [5], which is applicable in that a functional Ψ, differentiable with respect to a p-variation norm for $p < 2$, may have a Fréchet derivative $D\Psi$ such that $D\Psi(\alpha_n)$ converges in distribution to $D\Psi$ of a Brownian bridge, while the p-variation bounds a remainder term.

In another direction, Markov process paths in metric spaces, under rather general moment bounds on increments, have bounded p-variation for $p > 2$ [4]. Here one can consider for example Lévy processes in 3-dimensional hyperbolic (Lobachevsky) space, which is the space of possible velocities of bodies in special relativity. The resulting trajectories in space are not very rough paths, in that they are Lipschitz, but the velocity process can have jumps.

References

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